Lecture 7

13.1 - Vector Functions & Space Curves.

The vector equation for a line (or a line segment if the values of t are restricted) is an example of a vector function. In general, a vector function has the form

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$= f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}.$$

Terminology: t is called the parameter.

The domain of such a function is the "intersection" (points in common) of the domains of the coordinate functions f, g, and h.

Ex: What is the domain of $\vec{r}(t) = \langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \rangle$

Sol:

Function	Domain
$\sqrt{4-t^2}$	$[-2, 2]$
e^{-3t}	$\mathbb{R} = (-\infty, \infty)$
$\ln(t+1)$	$(-1, \infty)$

So, the domain of $\vec{r}$ is

$$D = [-2,2] \cap (-\infty, \infty) \cap (-1, \infty) = (-1, 2]$$

$\diamond$

Sometimes, instead of writing f, g, and h for the component function, we use $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ to help keep straight which function is in which component.

Limits: We would like to talk about the continuity of vector functions.

Recall that a function $f(t)$ is continuous at a if $\lim_{t \to a} f(t) = f(a)$.

Def: 1) The limit of a vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ as t approaches a is:

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

2) The vector function $\vec{r}(t)$ is continuous at a if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$

Let's look at some examples of vector functions:

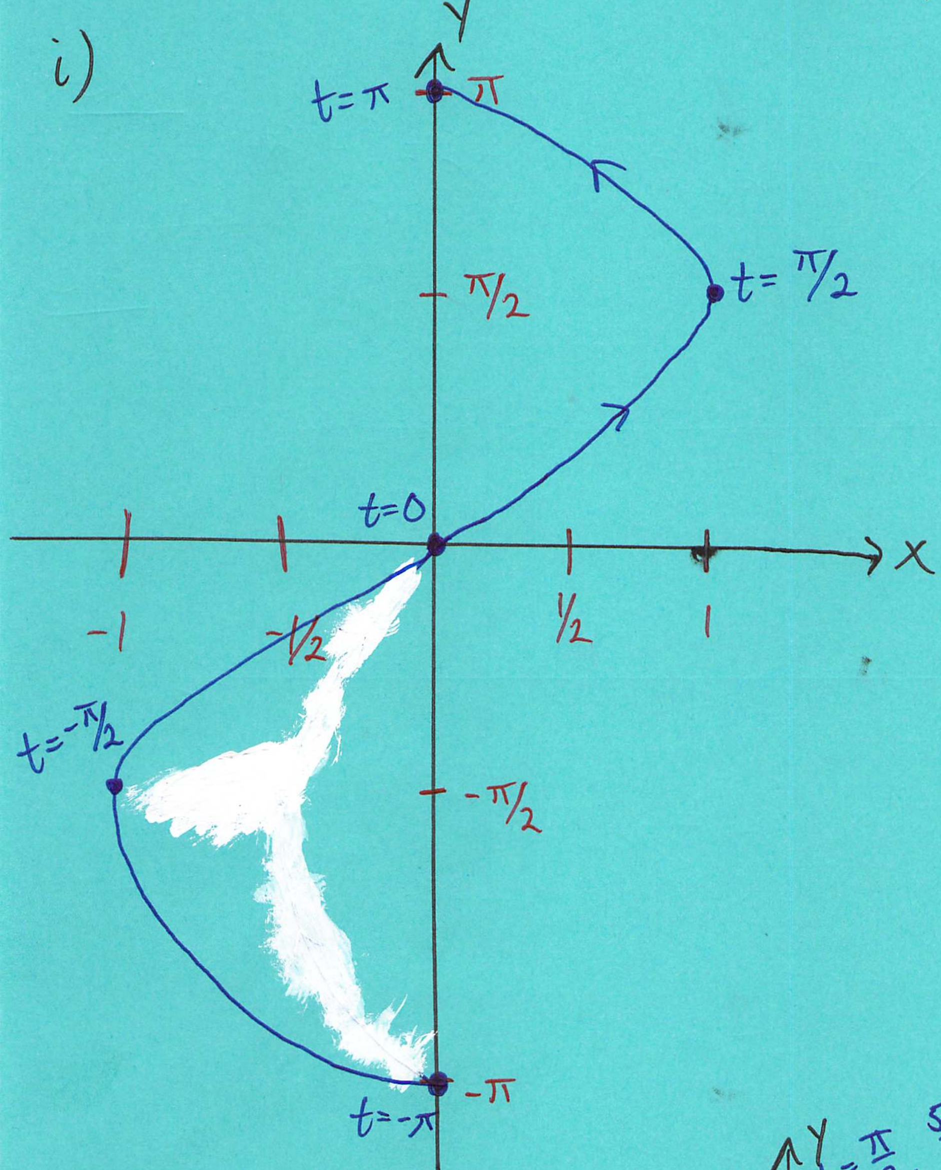
When sketching curves, it's useful to use arrows to indicate increasing t values.

<u>Ex</u>: i) $\vec{r}(t) = \langle \sin t, t \rangle$; ii) $\vec{r}(t) = \langle 3\cos t, \sin t \rangle$

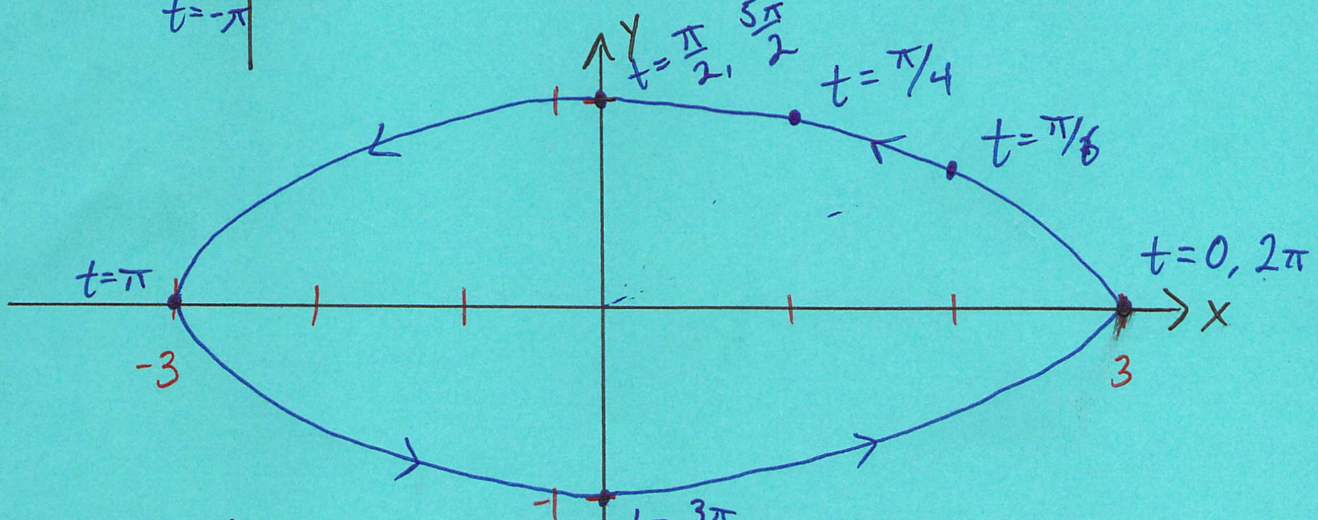
iii) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$; iv) $\vec{r}(t) = \langle t, \sin t, 2\cos t \rangle$

v) $\vec{r}(t) = \langle t\cos t, t\sin t, t \rangle$

i)



ii)



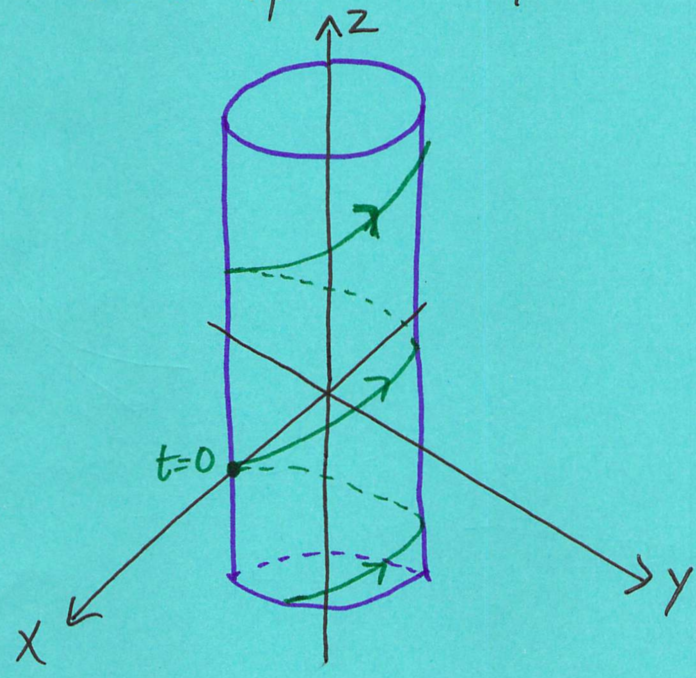
<u>Notice</u>: $x = 3\cos t$ $\quad \dfrac{x^2}{9} + y^2 = 1$ (eqn of an ellipse)

$y = \sin t$

iii) Sometimes, to visualize these curves, it is useful to see them as sitting on a surface and/or by looking at their projections to coordinate planes. Let's do both here.

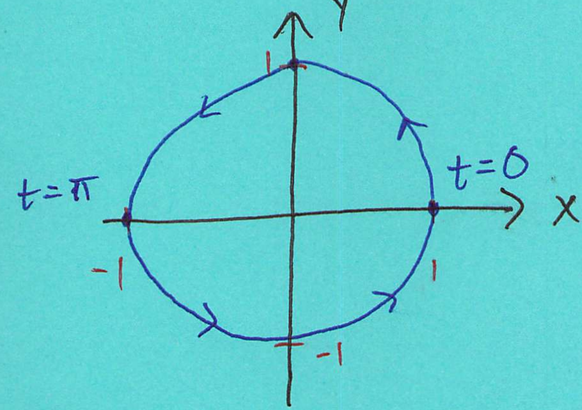
· <u>On Surfaces</u>: Notice that $\underset{x}{\underset{\shortparallel}{[f(t)]}}^2 + \underset{y}{\underset{\shortparallel}{[g(t)]}}^2 = \cos^2 t + \sin^2 t = 1$,

so the curve satisfies $x^2+y^2=1$, and so sits on that cylinder. Since the z-component is t, that just means we move up the cylinder as t increases:

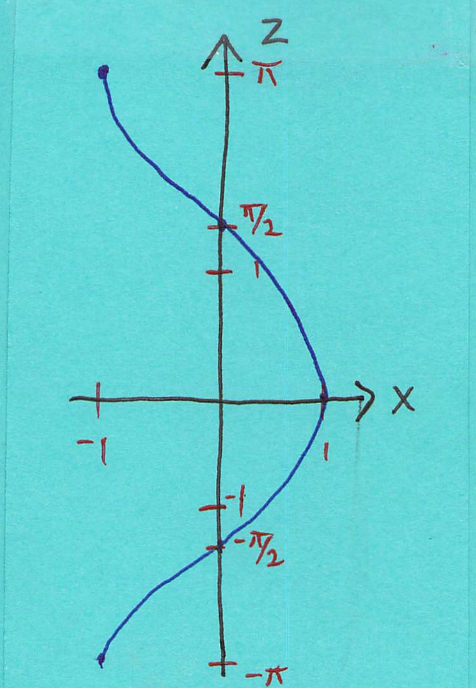


• With projections

Projection to: • xy-plane: $\vec{r}(t) = \langle \cos t, \sin t \rangle$



• xz-plane $\vec{r}(t) = \langle \cos t, t \rangle$



• yz-plane : $\vec{r}(t) = \langle \sin t, t \rangle$

(see example 1)

Putting these together, we can recreate the "spiral"!

iv) Notice $y=\sin t$ & $z=2\cos t$, so $y^2+\frac{z^2}{4}=1$

so the graph sits on the elliptical cylinder.

(See mathematica code for graph.)

v) Notice $x=t\cos t$, $y=t\sin t$, $z=t$ &

$$x^2+y^2=t^2(\cos^2 t+\sin^2 t)=t^2=z^2$$

So, the graph lies on the cone $x^2+y^2=z^2$.

(See Mathemathica code for graph.)

A fun example:

$$\vec{r}(t)=\langle (4+\sin 20t)\cos t, (4+\sin 20t)\sin t, \cos 20t\rangle$$

Next, we can ask about the curves two surfaces intersect along. For example, two planes intersect along a line. What about more general surfaces?

<u>Ex</u>: Find a vector function for the intersection of the surfaces $z=x^2$ & $x^2+y^2=4$.

Sol: There are many ways to go about this, but let's take one of the easiest. We know that the curve has to lie on both surfaces. So, if $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then we have

$$h(t) = [f(t)]^2 \ \& \ [f(t)]^2 + [g(t)]^2 = 4$$

To get the function, it's usually easiest to choose something for one component, and solve for the other two. Of course, it should be an intelligent choice. For example, $f(t)$ is in both equations, so let's choose something for $f(t)$. The second equation suggests taking $f(t) = 2\cos t$ (this is what I meant by choosing intelligently). Then we get $g(t) = 2\sin t$ & $h(t) = 4\cos^2 t$.

(See code for pictures.)

So, an equation for the intersection is:

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\cos^2 t \rangle$$